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The Time Equation of Satellite Orbit Theory

by

George B. Findley

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ABSTRACT

The time equation of satellite orbit theory is defined as the equation used to predict the time at which a satellite will either cross the earth's equator or pass through the perigee point of its orbit, depending, respectively, on whether the nodal or anomalistic period is used. This paper gives a derivation of the time equation through terms of order of the fourth time derivative of the period. It is shown that when the time equation is expressed as a polynomial the initial period is the sum of the coefficients of the variable. Since the time equation is also used to determine the numerical values of its coefficients by least square fitting the equation to observations, equations are derived for the first and second time derivatives in terms of these coefficients. An analysis of some NORAD and NSSCC data indicates, at least for certain satellites, that determining optimal values of these coefficients to use in the satellite position predicting method is closely related to the time interval over which the least square method is applied.

This technical documentary report has been reviewed and is approved.

ROBERT H. WARREN
Major General, USAF
Commander

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ILLUSTRATIONS AND TABLES

SECTION 1 - INTRODUCTION

The time equation of satellite orbit theory is defined as the equation used for predicting the time at which a satellite will either cross the earth's equator or pass through the perigee point of its orbit, depending, respectively, on whether the nodal or anomalistic period is used. This paper gives a derivation of the time equation through terms of order of the fourth time derivative of the period. By the use of Kepler's third law the time derivatives of the period can be replaced by time derivatives of the semi-major axis, but this is not carried out in the paper. It is shown that when the time equation is expressed as a polynomial the initial period is the sum of the coefficients of the variable. Since these coefficients are usually obtained by least square fitting the time equation to observations, the period and also the first and second time derivative of the period are expressed in terms of these coefficients.

SECTION 2 - THE TIME EQUATION

The time equation can be written in the forms (1)[1], (2), or (3).

$$T_N = T_0 + P_0 + P_1 + P_2 + \ldots + P_{N-1} \,. \tag{1}$$

$$T_N = T_0 + \sum_{j=1}^{\infty} A_j \cdot F_j\,(N). \tag{2}$$

$$T_N = T_0 + \sum_{j=1}^{\infty} B_j \cdot N^j. \tag{3}$$

We will call (1) the exact time equation, where P_{N-1} is the time required for a satellite to complete its Nth revolution since the epoch time T_0. If the periods P_j ($j = 0, 1, 2, \ldots, N-1$) are the nodal periods, then T_N is the time of the Nth (south to north) equatorial crossing. If P_j is the anomalistic period then T_N is the time at which the satellite passes through perigee.

[1]Figures in parentheses refer to equations.

Equation (2), while not in general use, is given in this paper because it serves as an intermediate equation in deriving (3) from (1), and also because the coefficients A_j represent a single L^{pqr}_{st} defined in equation (4) following. In (2), A_1 is the epoch period P_0.

Equation (3) is most generally used to obtain T_N. It has been used for several years at the National Space Surveillance Control Center (NSSCC), now the Space Track Research and Development Facility, L. G. Hanscom Field, Bedford, Massachusetts, and it has also been used at the North American Air Defense Command (NORAD), Colorado Springs, Colorado. Both NORAD and NSSCC have found that fairly accurate ephemerides, for most artificial earth satellites, can be issued when either the quadratic or cubic equation in N is used. The coefficients B_j in (3) are expressible as a sum of the L^{pqr}_{st} quantities defined in (4).

In Appendix I the Taylor series for P_N is used in conjunction with (1) to obtain the following expressions for $A_j \cdot F_j$ (N). To simplify the writing of these expressions we first define L^{pqr}_{st} by the equation

$$L^{pqr}_{st} = P_0^{\,p}\,\dot{P}_0^{\,q}\left[\frac{\ddot{P}_0}{2!}\right]^r\left[\frac{\dddot{P}_0}{3!}\right]^s\left[\frac{\ddddot{P}_0}{4!}\right]^t, \tag{4}$$

where the dots denote time derivatives. We see from (4) that t is either zero or one, and for the latter value it follows q = r = s = 0. The terms of (2) for j = 1, 2, 3, ..., 12 are given in (5) through (16).

$$F_1(N)\cdot A_1 = N\,L^{100}_{00}. \tag{5}$$

$$F_2(N)\cdot A_2 = \frac{N\,(N-1)}{2}\,L^{110}_{00}. \tag{6}$$

$$F_3(N)\cdot A_3 = \frac{N\,(N-1)\,(2N-1)}{6}\,L^{201}_{00}. \tag{7}$$

$$F_4(N)\cdot A_4 = \frac{N\,(N-1)\,(N-2)}{6}\,L^{120}_{00}. \tag{8}$$

$$F_5(N)\cdot A_5 = \frac{N^2(N-1)^2}{4}\,L^{300}_{10}. \tag{9}$$

$$F_6(N)\cdot A_6 = \frac{N\,(N-1)\,(N-2)\,(N-3)}{24}\,L^{130}_{00}\;. \tag{10}$$

$$F_7(N)\cdot A_7 = \frac{N(N-1)\,(N-2)\,(2N-1)}{6}\,L^{211}_{00}\;. \tag{11}$$

$$F_8(N)\cdot A_8 = \frac{N(N-1)\,(2N-1)\,(3N^2 - 3N-1)}{30}\,L^{400}_{01}\;. \tag{12}$$

$$F_9(N)\cdot A_9 = \frac{N\,(N-1)\,(N-2)\,(N-3)\,(N-4)}{120}\,L^{140}_{00}\;. \tag{13}$$

$$F_{10}(N)\cdot A_{10} = \frac{N\,(N-1)\,(N-2)\,(42N^2 - 39N-1)}{120}\,L^{310}_{10}\;. \tag{14}$$

$$F_{11}(N)\cdot A_{11} = \frac{N\,(N-1)\,(N-2)\,(22N^2 - 69N + 29)}{120}\,L^{221}_{00}\;. \tag{15}$$

$$F_{12}(N)\cdot A_{12} = \frac{N\,(N-1)\,(N-2)\,(16N^2 - 22N + 2)}{120}\,L^{302}_{00}\;. \tag{16}$$

In the L^{pqr}_{st} notation the highest power of N multiplying a given L^{pqr}_{st} is $p + q + r + s + t$. When the sums of these superscripts and subscripts are the same, as in (12) through (16), then numerical values given by these equations are under normal circumstances of about the same order of magnitude. Hence, if we wish to retain only terms through the third time derivative, then (12) through (16) can be omitted. If we retain only terms through order of the second time derivative, then (9) through (11) can also be omitted.

The coefficients in (3) are obtained by collecting like powers of N in (5) through (16), and are given in (17) through (21).

$$B_1 = L^{100}_{00} - \frac{L^{110}_{00}}{2} + \frac{L^{201}_{00}}{6} + \frac{L^{120}_{00}}{3} - \frac{L^{130}_{00}}{4} - \frac{L^{211}_{00}}{3} - \frac{L^{400}_{01}}{30}$$

$$+ \frac{L^{140}_{00}}{5} - \frac{L^{310}_{10}}{60} + \frac{29L^{221}_{00}}{60} + \frac{L^{302}_{00}}{30}, \tag{17}$$

$$B_2 = \frac{L_{00}^{110}}{2} - \frac{L_{00}^{201}}{2} - \frac{L_{00}^{120}}{2} + \frac{L_{10}^{300}}{4} + \frac{11L_{00}^{130}}{24} + \frac{7L_{00}^{211}}{6}$$

$$- \frac{5L_{00}^{140}}{12} - \frac{5L_{10}^{310}}{8} - \frac{15L_{00}^{221}}{8} - \frac{5L_{00}^{302}}{12}, \tag{18}$$

$$B_3 = \frac{L_{00}^{201}}{3} + \frac{L_{00}^{120}}{6} - \frac{L_{10}^{300}}{2} - \frac{L_{00}^{130}}{4} - \frac{7L_{00}^{211}}{6} + \frac{L_{01}^{400}}{3}$$

$$+ \frac{7L_{00}^{140}}{24} + \frac{5L_{10}^{310}}{3} + \frac{7L_{00}^{221}}{3} + \frac{5L_{00}^{302}}{6}, \tag{19}$$

$$B_4 = \frac{L_{10}^{300}}{4} + \frac{L_{00}^{130}}{24} + \frac{L_{00}^{211}}{3} - \frac{L_{01}^{400}}{2} - \frac{L_{00}^{140}}{12} - \frac{11L_{10}^{310}}{8}$$

$$- \frac{9}{8} L_{00}^{221} - \frac{7}{12} L_{00}^{302}, \tag{20}$$

$$B_5 = \frac{L_{01}^{400}}{5} + \frac{L_{00}^{140}}{120} + \frac{7L_{10}^{310}}{20} + \frac{11L_{00}^{221}}{60} + \frac{2L_{00}^{302}}{15}. \tag{21}$$

Equations (17) through (21) give the coefficients in (3). Assume that numerical values of these coefficients have been obtained for a satellite by least square fitting (3) to the observations. We first determine the period at epoch T_0 by inspection of (17) through (21) and find that, if terms of order of the s-1 time derivative of the periods are retained, we have

$$P_0 = \sum_{j=1}^{s} B_j \ (s = 2, 3, 4, 5). \tag{22}$$

We next determine the rate of change of the period $\dot{P}_0$ in terms of these coefficients. Inspection of (17) through (21) reveals that expressions for $\dot{P}_0$ become complicated quite rapidly as s increases. For s equal to 4 or 5, it is necessary to solve a cubic or biquadratic equation, respectively, in order to obtain $\dot{P}_0$. Since in most applications of (3), as at NORAD and NSSCC, s is set equal to 2 or 3, we shall derive only expressions for these values of s. For s = 2, we obtain from (17) and (18):

$$B_1 = P_0 - \frac{P_0 \dot{P}_0}{2}, \tag{23}$$

$$B_2 = \frac{P_0 \dot{P}_0}{2}, \tag{24}$$

and we obtain from (23) and (24):

$$P_0 = B_1 + B_2, \tag{25}$$

$$\dot{P}_0 = \frac{2B_2}{B_1 + B_2} = \frac{2B_2}{P_0}. \tag{26}$$

For s = 3, we have from (17), (18), and (19):

$$B_1 = P_0 - \frac{P_0 \dot{P}_0}{2} + \frac{P_0 \dot{P}_0^2}{3} + \frac{P_0^2 \ddot{P}_0}{12}, \tag{27}$$

$$B_2 = \frac{P_0 \dot{P}_0}{2} - \frac{P_0 \dot{P}_0^2}{2} - \frac{P_0^2 \ddot{P}_0}{4}, \tag{28}$$

$$B_3 = \frac{P_0 \dot{P}_0^2}{6} + \frac{P_0^2 \ddot{P}_0}{6}. \tag{29}$$

From (27) through (29) we obtain P_0, $\dot{P}_0$, $\ddot{P}_0$, and, since these quantities should be obtained in that order, the following equations are given both in terms of the B's and of the previously calculated quantities.

For s = 3, we have:

$$P_0 = B_1 + B_2 + B_3, \tag{30}$$

$$\dot{P}_0 = 1 - \left[\frac{B_1 - 3B_2 - 5B_3}{B_1 + B_2 + B_3}\right]^{1/2}$$

$$= 1 - \left[1 - \frac{4B_2 + 6B_3}{P_0}\right]^{1/2}, \tag{31}$$

$$\ddot{P}_0 = \frac{10B_3 + 2B_2 - 2B_1}{(B_1 + B_2 + B_3)^2} + \frac{2(B_1 - 3B_2 - 5B_3)^{1/2}}{(B_1 + B_2 + B_3)^{3/2}}$$

$$= \frac{6B_3}{P_0{}^2} - \frac{\dot{P}_0{}^2}{P_0}. \tag{32}$$

In general, sufficient accuracy of $\dot{P}_0$ is obtained if (31) is expanded into an infinite series and only the first term is retained.

$$\dot{P}_0 \approx \frac{2B_2 + 3B_3}{P_0}. \tag{33}$$

SECTION 3 - THE TIME EQUATION AS USED AT NORAD AND NSSCC

The method of predicting satellite positions which was, until recently, used at NORAD and NSSCC was originally formulated by NSSCC. In this paper the NORAD/NSSCC method, or simply method, refers to the method used until recently. Since the beginning of 1962, improvements have been introduced at SPADATS which include a differential correction sequence in the predicting method and which makes the system semi-automatic in its use.

In April 1961, the author obtained from NSSCC the IBM orbital element cards for all satellites in orbit from January 1959 to April 1961. The elements given on these cards are nodal elements. These IBM cards contain the orbital elements used to obtain the Space Track bulletins (ephemerides) issued during this time interval. The bulletins were generally issued to cover a time interval of 3 to 10 days. One of the purposes for obtaining these cards was to undertake an error analysis in the hope that this analysis would shed some light on possible ways of improving predictions. While improvements appear to be definitely possible, at least for certain satellites, the author believes that a complete proof of such improvements must be verified with the observational data at NORAD or NSSCC. Therefore, the author has made arrangements to secure the necessary observational data and reduction program from NSSCC.

If improvements are to be made in the NORAD/NSSCC method, one should first determine what appears to be the major cause of correctable errors, and then determine if a partial or complete elimination of this cause leads to better predictions. Before pointing out what appears to be the major cause of errors in predictions, some opinions on other causes of errors will be given. First, the author believes the observational data received at NORAD and NSSCC are sufficiently accurate to permit more precise predictions. Secondly, the equations employed in the NORAD/NSSCC method are sufficiently accurate to permit more precise predictions, and it is therefore not necessary to include more theoretical terms in these equations until such times as errors in predictions are explainable by their omission. Thirdly, while unpredictable variations in air drag are certainly a serious drawback to long range predictions, or sometimes even to short range predictions when rapid acceleration variations occur due to geomagnetic disturbances (such as the October and November 1960 events [1][2]), it is possible to make longer range predictions and still remain within the limit of permissible errors.

The author believes, and knows that a good many people in NSSCC and NORAD concur, that the major causes of errors in predicting satellite positions are errors in the coefficients of the time equations. In order to point out the importance of these coefficients, the first three equations used in the NORAD/NSSCC method are listed below. Since in practice it is generally necessary to retain the cubic term in the time equation only during the decay phase of a satellite, we shall therefore drop this term. We employ the notation used at NORAD and NSSCC.

$$T_N = T_0 + (N - N_0) P_0 + (N - N_0)^2 C_0. \tag{34}$$

$$\Omega_N = \Omega_0 + (T_N - T_0) \dot{\Omega}_0 + (T_N - T_0)^2 \frac{\ddot{\Omega}_0}{2}. \tag{35}$$

$$\omega_N = \omega_0 + (T_N - T_0) \dot{\omega}_0 + (T_N - T_0)^2 \frac{\ddot{\omega}_0}{2}. \tag{36}$$

The only assumptions in these three equations are that the period, the right ascension of the ascending node Ω, and the argument of perigee ω can be adequately represented by their Taylor series. This assumption on the period is used in the appendix to derive the time equation. In (34), P_0 is called the nodal period at revolution number N_0, and $C_0 = \frac{P_0 \dot{P}_0}{2}$. While it is true, as pointed out in the previous section, that

[2]Numbers in brackets refer to references.

when the quadratic equation is used the actual nodal period at revolution N_0 should be assigned the value of $P_0 + C_0$, the assignment of this value will contribute nothing to the accuracy of predictions unless the error in P_0 is less than the magnitude of C_0. In general this is not the case. However, if (34) is fitted to observations and the P_0 and C_0 are obtained to such accuracy that the magnitude of C_0 affects the accuracy of the initial period, then the nodal period should be assigned the value $P_0 + C_0$, and this value used in computing the anamolistic period, the semi-major axis, etc. Of course, values of P_0 obtained by the least square fitting of (34) to the observations are to be retained in (34), and we must never replace P_0 by $P_0 + C_0$ in the time equation. In what follows we shall always refer to P_0 in (34) as the initial period at revolution number N_0.

In predicting future positions of a satellite, (34) is first used to obtain T_N, and then $T_N - T_0$ is substituted into (35) and (36) to obtain Ω_N and ω_N. Hence, it is readily seen that any errors in the time equation produce errors in subsequent computations in the predicting method. It is therefore essential that the time equation be made to predict as accurately as possible.

In order to understand how errors are introduced into the coefficients of the time equation, we shall point out how the coefficients of the time equation are usually obtained. Let us first rewrite the time equation in the form,

$$T\ (N + N_i) = T\ (N_i) + N\ P\ (N_i) + N^2\ C\ (N_i). \tag{37}$$

Assume that at epoch revolution N_i the T, P, and C on the right hand side of (37) are known. Also assume that we restrict the time residuals $\Delta T\ (N + N_i)$ (differences between observed and predicted equatorial crossings) to a maximum absolute value of 30 seconds. The dots in Fig. 1 represent the time residuals ΔT. After N revolutions ($N = N_j - N_i$) it becomes apparent that, unless corrections are made in the time equation, the value of ΔT will exceed 30 seconds by revolution number N_k. Hence, it is imperative that a new Space Track bulletin be issued. The least square method is then used to fit the time equation to the observations, let us say over the time interval $T(N_j) - T(N_i)$, and a new set of coefficients obtained for (37). If the epoch revolution N_j is selected for the next bulletin, then to conform to the notation used in (37), N_i must be replaced by N_j in (37). Inherent in the application of the least square method, as applied to obtain the new time equation at N_j, is the expectation that the time residuals will continue to lie along the solid curve in Fig. 1 if the predictions are allowed to continue. If the time residuals did continue to lie along this curve, then we should expect accurate predictions when the coefficients at N_j are used in the time equation.

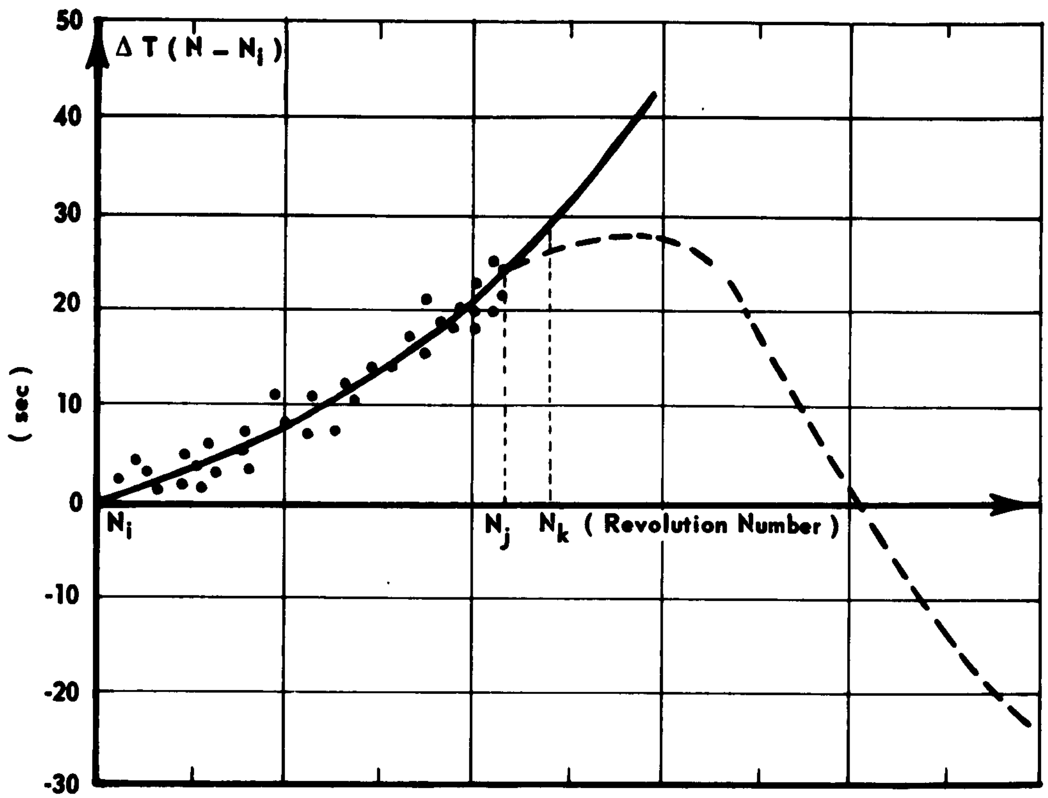

Fig. 1: Time Residuals Versus Revolution Number

However, in many cases the time residuals do not continue to lie along this curve, and in these cases the least square method does not give us a set of optimal coefficients. The time residuals could, in fact, continue along the dashed curve and actually reach a negative maximum permissible error before reaching a positive maximum permissible error. A serious dilemma is then encountered in the problem of making more accurate predictions, since we are faced with the almost impossible task of determining along which curve the time residuals will lie.

The author is currently investigating a number of methods which, for certain satellites, give more precise predictions. In order to describe one of these methods, suppose that the time equation (37) has been least square fitted to the incoming observations over the time interval $T(N_j) - T(N_i)$, and we then use (37) with N_i replaced by N_j to predict future nodal crossings. Why was the time interval $T(N_j) - T(N_i)$ used to obtain the orbital elements at $T(N_j)$ rather than some other time interval? The answer, of course, is that the orbital elements at $T(N_i)$ were used over this time interval to obtain the time residuals, and we have no other time interval to use. This answer leads to the following question. Is it

possible, for certain satellites, that the time equation should be least square fitted over a much longer time interval in order to obtain more accurate coefficients and hence minimize the time residuals? The author does not have available, at the present time, the necessary data to completely answer this question. However, it is possible to approximate the coefficients which would be obtained by least square fitting over a longer time interval by using the epoch revolutions and epoch times on three successive sets of orbital element cards. As an example of this procedure we use Space Track Object No. 63, 1960 Pi 1, Tiros II. From the three successive sets of orbital element cards issued at epoch revolutions 560, 650, and 910, we obtain the respective epoch times T(560), T(650), and T(910). These data, in conjunction with the time equation

$$T\ (N + N_i) = T\ (N_i) + N\ P\ (N_i) + N^2\ C\ (N_i), \tag{38}$$

give the two equations

$$T(560) = T(910) - 350\ P(910) + (350)^2\ C(910) \text{ and} \tag{39}$$

$$T(650) = T(910) - 260\ P(910) + (260)^2\ C(910), \tag{40}$$

which can be solved for P(910) and C(910). The equation

$$T(N + 910) = T(910) + N\ P(910) + N^2\ C(910) \tag{41}$$

is then used to predict the epoch times on later sets of orbital element cards. Before using (41) to predict these later epochs, we set $N = -470$ and predict the epoch T(440). The difference between this predicted epoch and the epoch on the orbital element cards is three seconds, and this indicates that the values of P and C in (41) should permit fairly precise predictions of future epochs. The results of such predictions are given in Fig. 2, where the ΔT curve gives the time residuals which would be obtained if the P and C given on the orbital element cards issued at revolution number 910 were used in (41), and the $\overline{\Delta T}$ curve gives the time residuals which would be obtained if (39) and (40) are solved for P and C and these values used in (41).

The decision as to whether a longer time interval should be used for a specific satellite should certainly not rest on a single example as shown in Fig. 2. This decision should be made on the consistency of improvements. For Tiros II, when the method was applied to 14 consecutive sets of cards, including the sets issued at epoch revolution 910, in each application we had $\overline{\Delta T} < \Delta T$. For another example we use Space Track Object No. 29, 1960 Beta 2, Tiros I. In Fig. 3 the sets of cards issued

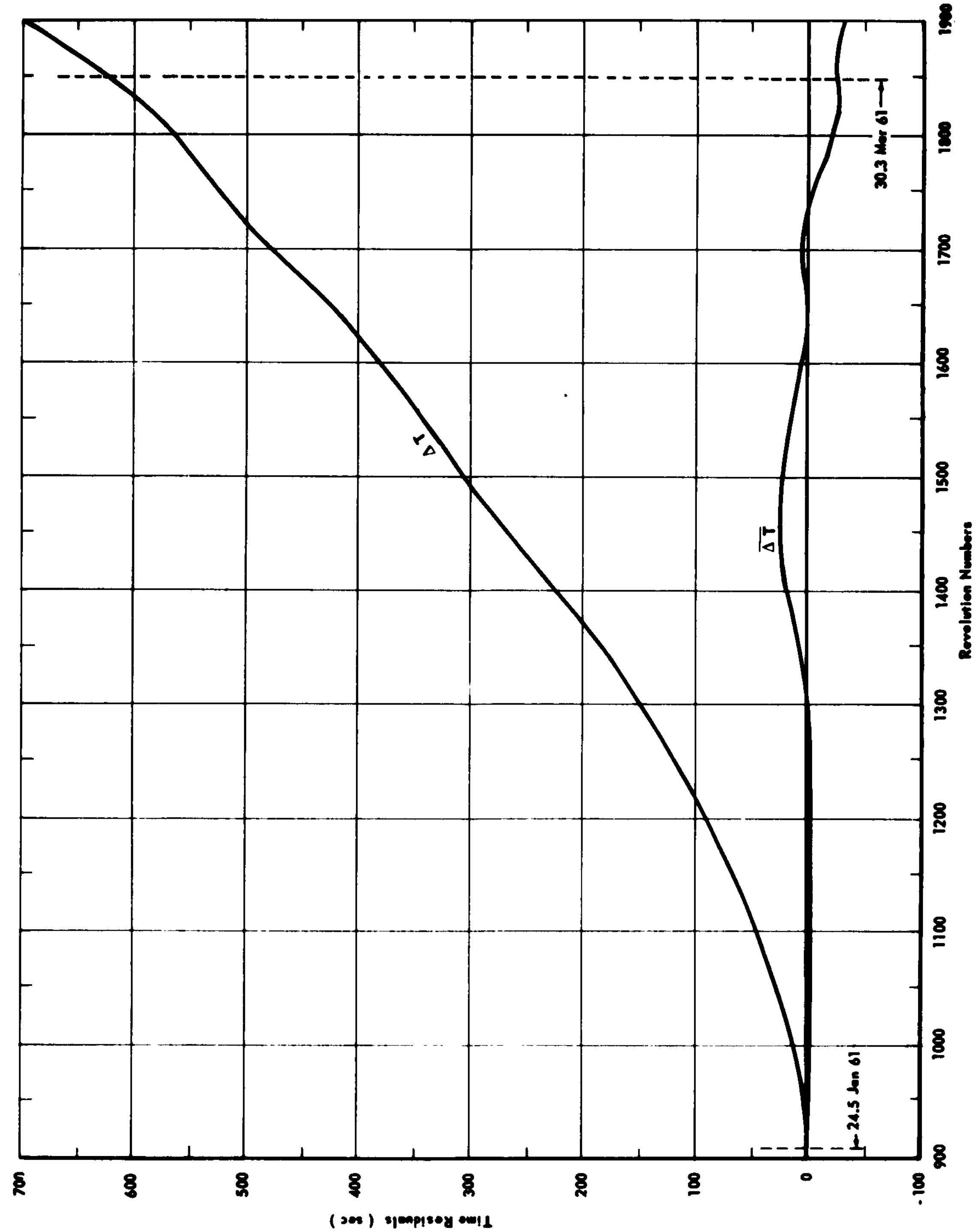

Fig. 2: Satellite 1960 Pi 1, Tiros II

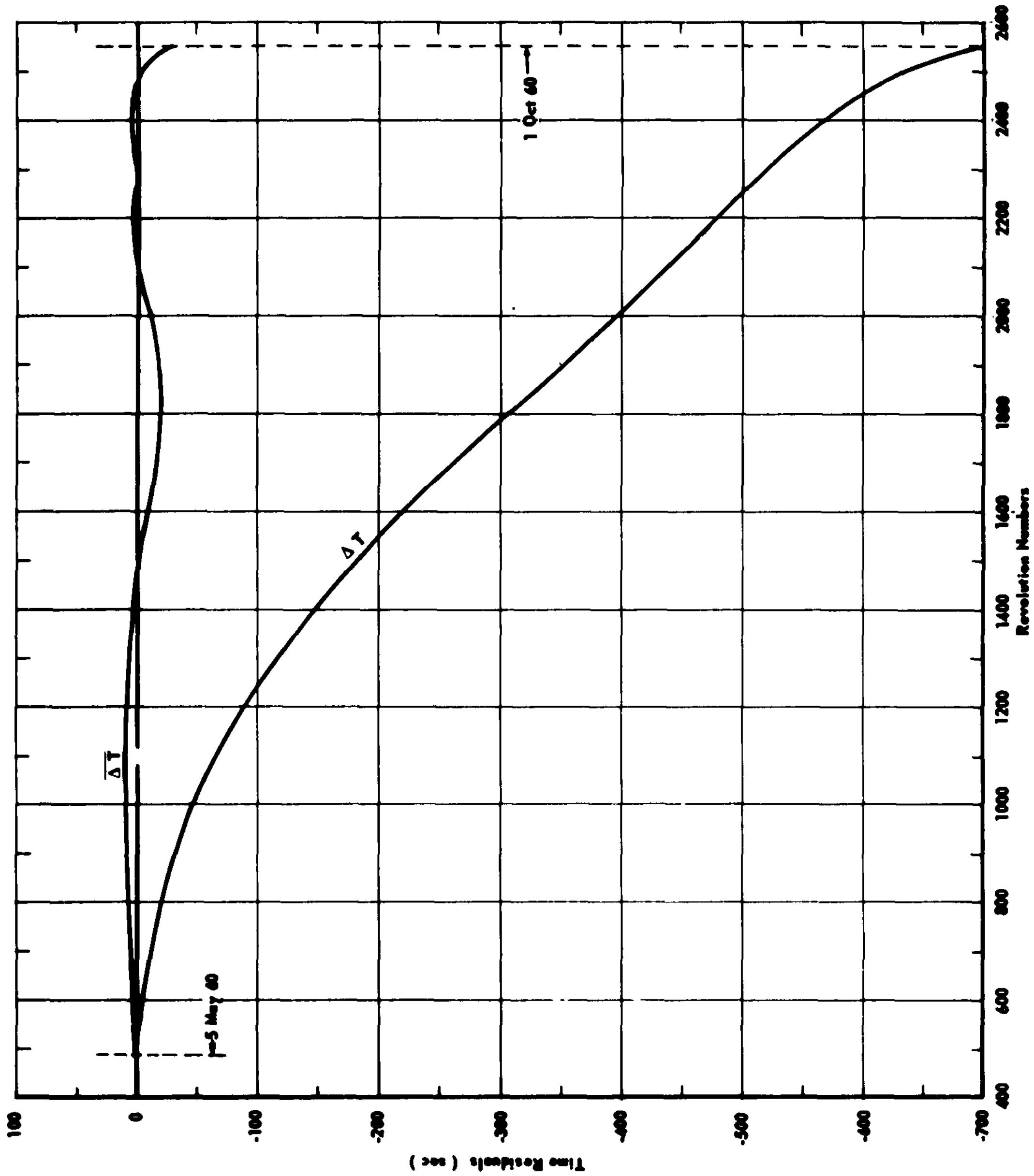

Fig. 3: Satellite 1960 Beta 2, Tiros I

at epoch revolution numbers 100, 250, and 490 were used to obtain the coefficients of the time equation at 490. In Fig. 4, the sets of cards issued at revolution numbers 2500, 2650, and 2970 were used to obtain the coefficients of the time equation at 2970. In Figs. 3 and 4, the ΔT and $\overline{\Delta T}$ notation is the same as in Fig. 2.

A comparison of the ΔT and $\overline{\Delta T}$ curves in Figs. 2 through 4 indicates that, for certain satellites, the accuracy and also the time intervals over which Space Track bulletins can be extended are dependent on how much of a satellite's recent history is incorporated into determining the coefficients of the time equation.

In order to further point out that it is the coefficients of the time equation which are critical in accurate position predictions, we use data from Space Track Object No. 28, 1960 Beta 1, rocket body of Tiros I. Starting at epoch revolution 250, the equation

$$\omega_N = \omega_{250} + (T_N - T_{250})\,\dot{\omega}_{250} + (T_N - T_{250})^2 \frac{\ddot{\omega}_{250}}{2}, \tag{42}$$

with $T_{250} = 109.69590$; $\omega_{250} = 185.7047$; $\dot{\omega}_{250} = 4.117376$;

$\frac{\ddot{\omega}_{250}}{2} = 9.34 \times 10^{-6}$, predicts that values of ω_N on the successive 29 sets of orbital element cards with a maximum error less than 0.025 degree. In making these predictions the values of T_N on the respective cards were used. These cards cover a time interval of 351 days, or 5100 revolutions, and during this time the perigee has moved through an angle of 1446.5 degrees. The accuracy of (36), when applied to 1960 Beta 1, is not to be construed as indicative of the accuracy with which (36) will predict the argument of perigee for other satellites over long time intervals. For example, when (36) is least square fitted to 1958 Alpha data (see [2]) over a time interval of approximately two years, the maximum ω residuals are approximately two degrees. The example cited here, 1958 Beta 1, is solely for the purpose of pointing out that while other equations employed in the NORAD/NSSCC method are sufficiently accurate for longer range predictions, this fact is of little value unless one can predict with sufficient accuracy the value of T_N. In obtaining the $\overline{\Delta T}$ curve in Fig. 5 the coefficients of the time equation at epoch revolution 490 were obtained by using the epoch times given on successive sets of orbital element cards issued at revolution numbers 100, 250, and 490. The ΔT curve was obtained by using the P and C given on the cards issued at epoch revolution 490.

Inspection of the $\overline{\Delta T}$ curve reveals that the maximum error in predicting 4860 revolutions into the future is probably about 240 seconds. While this maximum error is approximately eight times our assumed

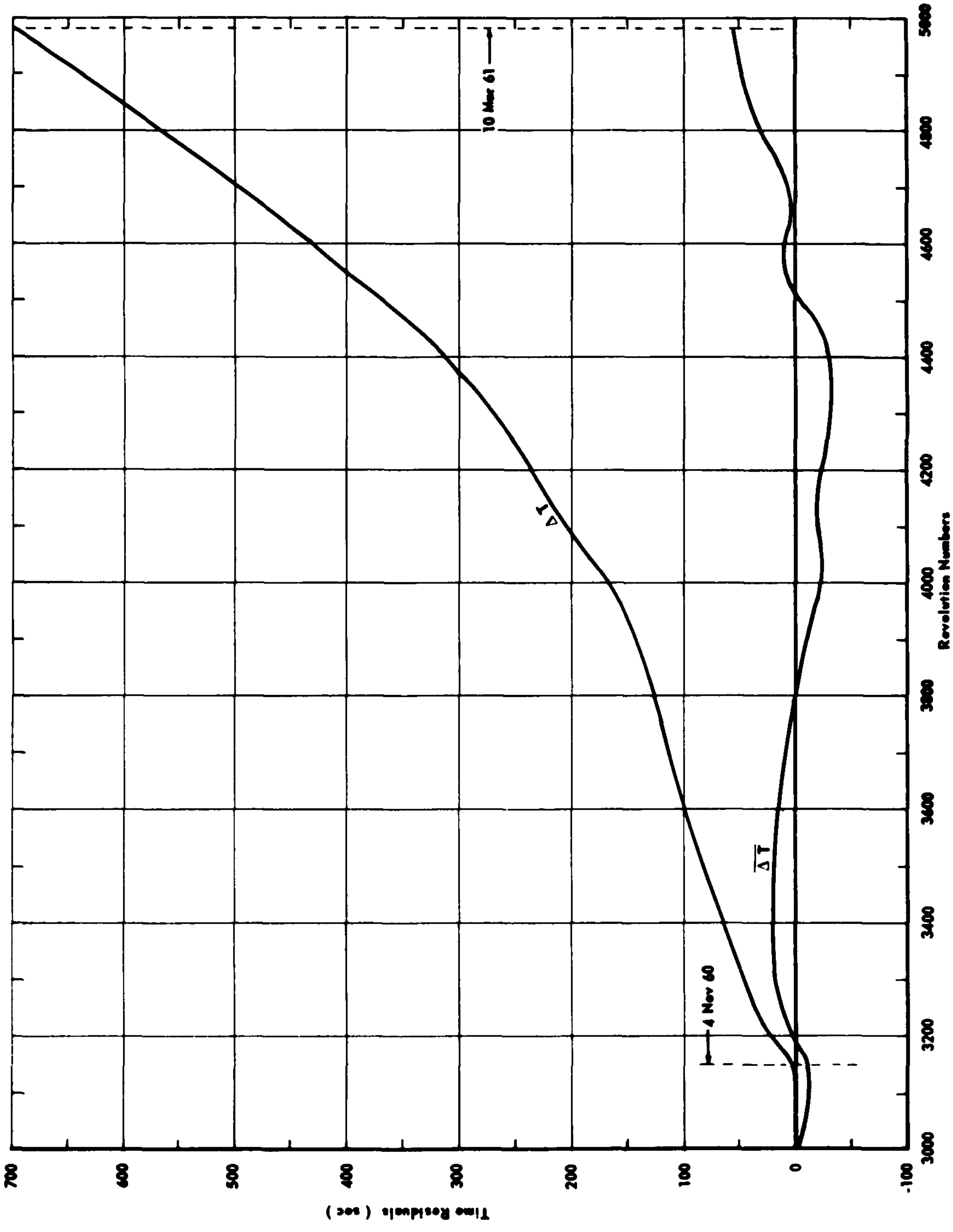

Fig. 4: Satellite 1960 Beta 2, Tiros I

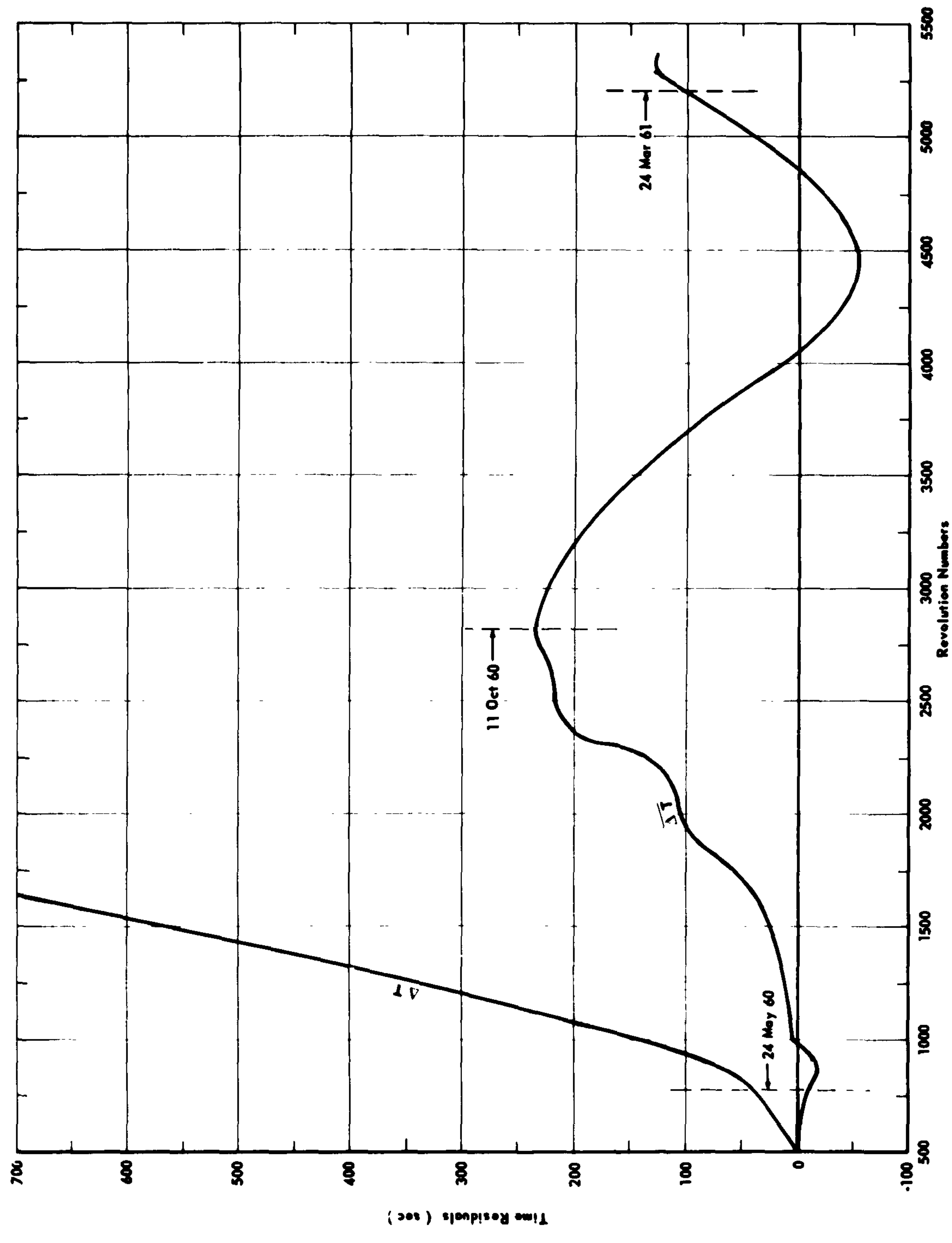

Fig. 5: Satellite 1960 Beta 1 - Rocket Body of Tiros I

permissible time error, the use of these predicted T_N values in (42), instead of the T_N values on the orbital element cards, will only change the predicted ω_N values by approximately 0.01 degree. This shows that whereas errors in time may be excessive, such errors do not necessarily introduce serious errors into the angles.

It should be strongly pointed out that the method of obtaining the coefficients of the time equation, in order to obtain the $\overline{\Delta T}$ values given in Figs. 2 through 5, is not recommended by the author for operational use, because the coefficients obtained in this manner are generally not the optimal values (see Fig. 6). What is recommended, however, is that, in order to make more accurate and longer range predictions, serious consideration be given to the time interval over which the least square method is applied.

The ΔT curve in Fig. 6 for Space Track Object No. 36, 1960 Epsilon 3 (metal object from Sputnik IV), was obtained by using in the time equation the P and C on the orbital element cards issued at epoch revolution number 4440. The $\overline{\Delta T}$ curve was obtained when the three successive sets of orbital element cards issued at revolution numbers 4150, 4280, and 4440 were used to calculate the P and C at revolution 4440. While $\overline{\Delta T} < \Delta T$ in Fig. 6, the $\overline{\Delta T}$ curve is certainly not the optimal time residual curve. If the period P used in obtaining the $\overline{\Delta T}$ curve is changes by 0.25 second, the $\overline{\overline{\Delta T}}$ curve is obtained. The $\overline{\overline{\Delta T}}$ curve is approximately a sine curve with a period of 27 days and an amplitude of 12 seconds.

The $\overline{\overline{\Delta T}}$ curve can be considered as an empirical curve which is drawn after the time residuals are known. However, if the criteria (referred to in the following section) are applied to the P and C values used in the $\overline{\Delta T}$ curve, the cubic term in the time equation will only account for approximately five seconds of the error at the end of the $\overline{\Delta T}$ curve. Hence, since we are justified in using the quadratic time equation over this time interval, the assumption that the residuals are caused by errors in the P and C values and by the omission of periodic terms leads to correcting the P by 0.25 second, and gives the best fit curve which is the $\overline{\overline{\Delta T}}$ curve.

SECTION 4 - COMMENTS

In Section 2 and Appendix I, the time equation was derived to include all secular terms of order of the fourth time derivative of the period. The

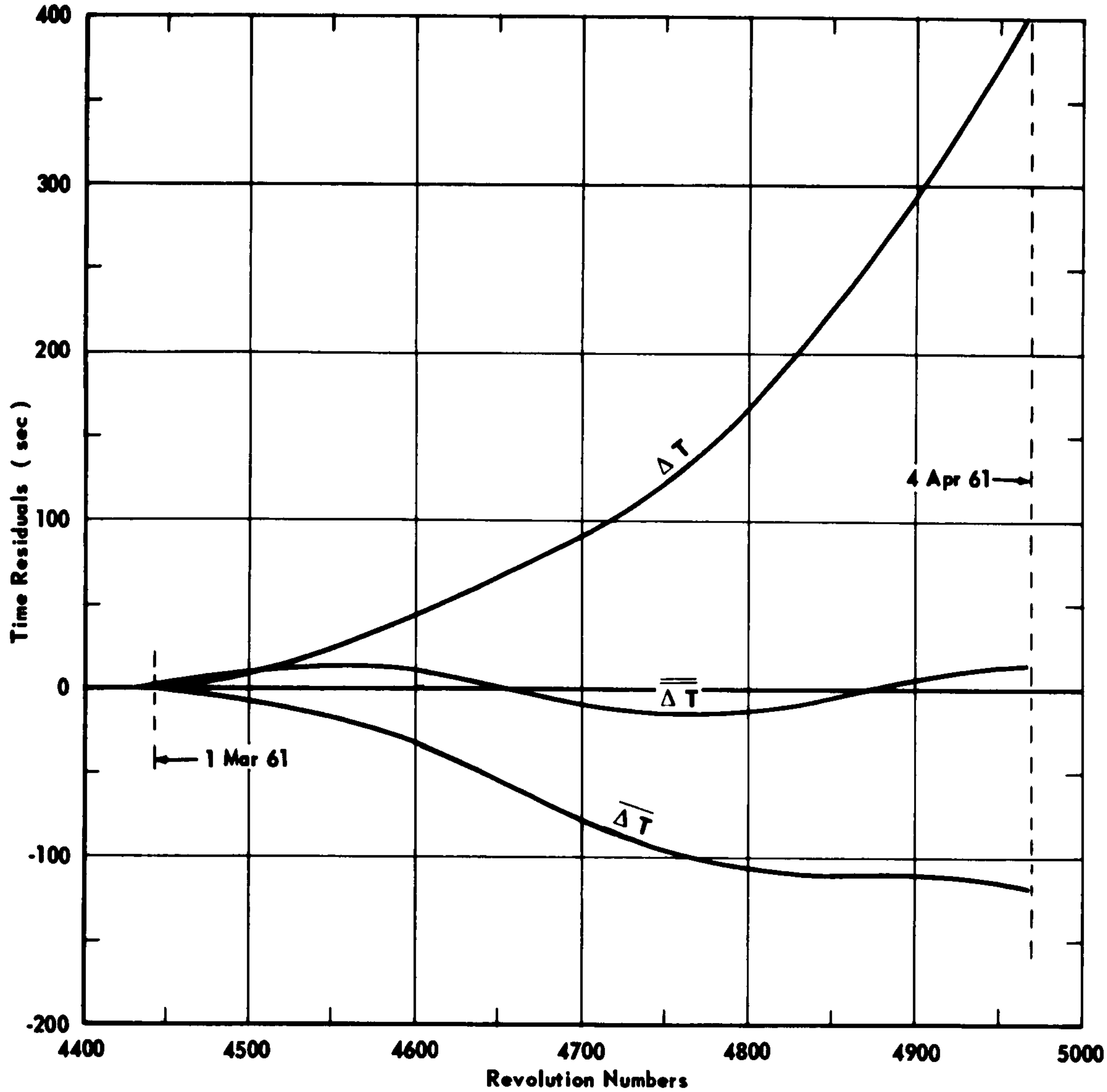

Fig. 6: Satellite 1960 Epsilon 3 - Metal Object from Sputnik IV

previous sections could leave the reader with the impression that it is only P and P that are important in the time equation, and this is certainly not the case. The terms containing $\ddot{P}$ through $\overset{\cdot\cdot\cdot\cdot\cdot}{P}$ in (17) through (21) are important in making an accurate error analysis of applications of the time equation. Due to the fact that accurate expressions for $\ddot{P}$ through $\overset{\cdot\cdot\cdot\cdot\cdot}{P}$ can be derived in terms of P and $\dot{P}$, the author has developed criteria for determining good approximations to the optimal time interval over which the time equation should be least square fitted. Applications of these equations have increased the accuracy of predicting satellite positions and have also

improved satellite decay predictions. In addition to P and $\dot{P}$, it may sometimes be necessary to include the eccentricity and its first time derivative in these equations. The author is continuing to develop and evaluate these equations and expects to publish the results at a later date.

In deriving the time equation, periodic terms were intentionally omitted because they were not included in the NORAD/NSSCC method. Their omission is entirely justifiable because the residuals are primarily due to inaccuracies of the secular terms and to variations in atmospheric density. Actually, the residuals, once the bad errors in secular terms are eliminated, do show up in nearly all the curves given in Section 3. The decision, therefore, on whether to include or not include the periodic terms depends primarily on the accuracy to which the secular terms have been obtained. For most orbiting satellites the inclusion of periodic terms in the NORAD/NSSCC method should not be an exceedingly difficult task.

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APPENDIX I

DERIVATION OF THE TIME EQUATION

In this appendix the time equation is derived to include all terms of order of the fourth time derivative of the period. The derivation applies both to nodal and anomalistic periods as defined previously. The period P_N is expressible as a continuous function of time T_N,

$$P_N = P\,(T_N), \tag{1-1}$$

and the Taylor series for the period is

$$P_N = P_0 + [T_N - T_0]\,\dot{P}_0 + [T_N - T_0]^2\,\frac{\ddot{P}_0}{2!} + [T_N - T_0]^3\,\frac{\dddot{P}_0}{3!} + [T_N - T_0]^4\,\frac{\ddddot{P}_0}{4!} + \ldots. \tag{1-2}$$

The exact time equation is

$$T_N = T_0 + P_0 + P_1 + P_2 + \ldots + P_{N-1}, \tag{1-3}$$

where P_{N-1} is the time required for a satellite to complete the Nth revolution. From (1-3) we have

$$T_1 - T_0 = P_0. \tag{1-4}$$

Substitution of (1-4) into (1-2) gives P_1, and substitution of this value of P_1 into (1-3) gives

$$T_2 - T_0 = 2P_0 + P_0\dot{P}_0 + \frac{P_0^{\,2}\ddot{P}_0}{2!} + \frac{P_0^{\,3}\dddot{P}_0}{3!} + \frac{P_0^{\,4}\ddddot{P}_0}{4!}. \tag{1-5}$$

Equation (1-5) is now substituted into (1-2) to obtain P_2. It is necessary to repeat the above procedure until P_9 is obtained. Since the calculations become lengthy, the results are given in the following tabulated form, where

$$L^{pqr}_{st} = P_0^{\,p}\,\dot{P}_0^{\,q}\left[\frac{\ddot{P}_0}{2!}\right]^r\left[\frac{\dddot{P}_0}{3!}\right]^s\left[\frac{\ddddot{P}_0}{4!}\right]^t. \tag{1-6}$$

TABLE 1. COEFFICIENTS OF L_{st}^{pqr} FOR SUCCESSIVE PERIODS OF A SATELLITE

$T_N = T_0$	L_{00}^{100}	L_{00}^{110}	L_{00}^{201}	L_{10}^{300}	L_{01}^{400}	L_{00}^{120}	L_{00}^{130}	L_{00}^{140}	L_{00}^{211}	L_{10}^{310}	L_{00}^{221}	L_{00}^{302}
$+P_0$	1	0	0	0	0	0	0	0	0	0	0	0
$+P_1$	1	1	1	1	1	0	0	0	0	0	0	0
$+P_2$	1	2	4	8	16	1	0	0	5	13	1	4
$+P_3$	1	3	9	27	81	3	1	0	23	90	20	30
$+P_4$	1	4	16	64	256	6	4	1	62	324	96	112
$+P_5$	1	5	25	125	625	10	10	5	130	850	290	300
$+P_6$	1	6	36	216	1296	15	20	15	235	1845	685	660
$+P_7$	1	7	49	343	2401	21	35	35	385	3528	1386	1274
$+P_8$	1	8	64	512	4096	28	56	70	588	6160	2520	2240
$+P_9$	1	9	81	729	6561	36	84	126	852	10,044	4236	3672
+...												
$+P_{N-1}$												

The numbers in the jth row are the coefficients of the L^{pqr}_{st} quantities for the P_j in that row. The expressions for the coefficients of the L^{pqr}_{st} quantities for P_{N-1} are easily obtainable as functions of N but are unnecessary for our purpose. Inspection of the table shows that the sum of the numbers in the second column is N, and therefore the coefficient of L^{100}_{00} in the time equation is N. In the third column we have $1 + 2 + 3 + \ldots + (N-1)$, and therefore the coefficient of L^{110}_{00} is $\frac{N(N-1)}{2}$. In the fourth column we have $1^2 + 2^2 + 3^2 + \ldots + (N-1)^2$, hence the coefficient of L^{201}_{00} is $\frac{N(N-1)(2N-1)}{6}$. In the fifth column we have $1^3 + 2^3 + 3^3 + \ldots + (N-1)^3$, and therefore the coefficient of L^{300}_{10} is $\frac{N^2(N-1)^2}{4}$. In the sixth column we have $1^4 + 2^4 + 3^4 + \ldots + (N-1)^4$, and the coefficient of L^{400}_{01} becomes $\frac{N(N-1)(2N-1)(3N^2-3N-1)}{30}$.

The coefficients of the remaining seven L^{pqr}_{st} quantities are obtainable in the following manner. Since it was only necessary to obtain P_9 in order to derive the coefficient of L^{140}_{00}, we shall illustrate the method by obtaining the coefficient of L^{140}_{00}. Applying the method of differences to column 9 of Table 1, we have:

$$\begin{array}{ccccccccccc}
1 & & 5 & & 15 & & 35 & & 70 & & 126 \\
& 4 & & 10 & & 20 & & 35 & & 56 & \\
& & 6 & & 10 & & 15 & & 21 & & \\
& & & 4 & & 5 & & 6 & & & \\
& & & & 1 & & 1 & & & &
\end{array}$$

and therefore the sum of N numbers in the ninth column is

$$S_N = N + \frac{4N(N-1)}{2!} + \frac{6N(N-1)(N-2)}{3!} + \frac{4N(N-1)(N-2)(N-3)}{4!} + \frac{N(N-1)(N-2)(N-3)(N-4)}{5!}. \qquad (1\text{-}7)$$

Since there are N-4 numbers in the column for L_{00}^{140}, we replace N in (1-7) by N-4. When this substitution is made the coefficient of L_{00}^{140} is found to be

$$F_9(N) = \frac{N(N-1)(N-2)(N-3)(N-4)}{120}. \tag{1-8}$$

The remaining coefficients are found in a similar manner, and all 12 of the coefficients are given in (5) through (16) in Section 2.

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